Fundamentals of Physics II

Faculty of Physics-Kharazmi University

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اگر همواره مانند گذشته بیندیشید، همیشه همان چیزهایی را به‌دست می‌آورید که تاکنون کسب کرده‌اید

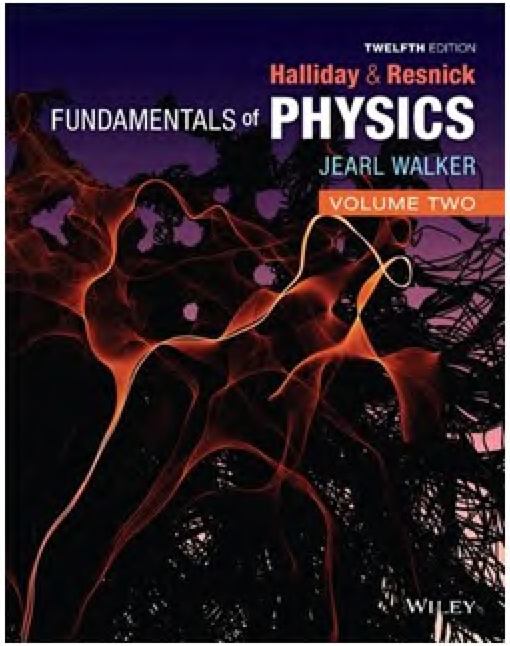
If you always think the way you've always thought, you'll always get what you've always got.

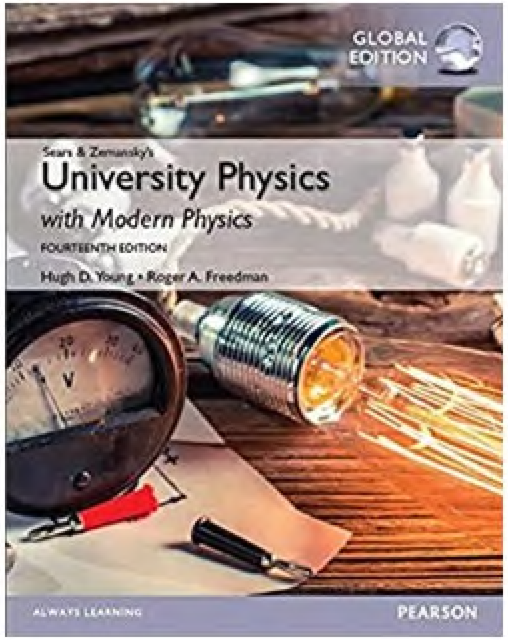
Fundamentals of Physics (12th Ed.)
Halliday, David; Resnick, Robert; Walker, Jearl

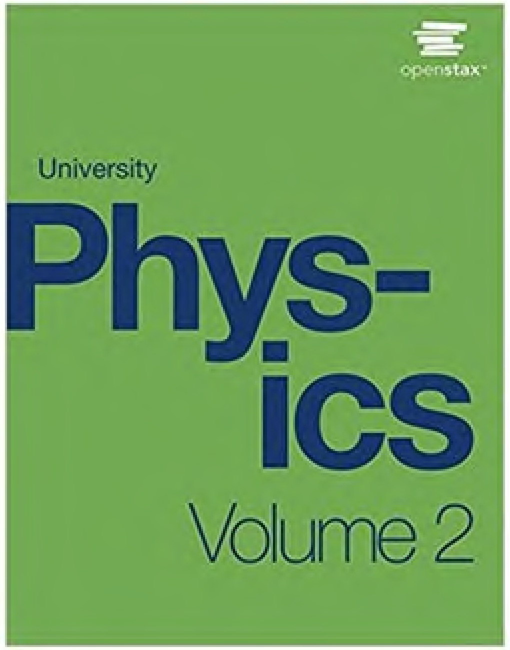
University Physics with Modern Physics (14th Global Ed.)
Hugh D. Young, Roger A. Freedman

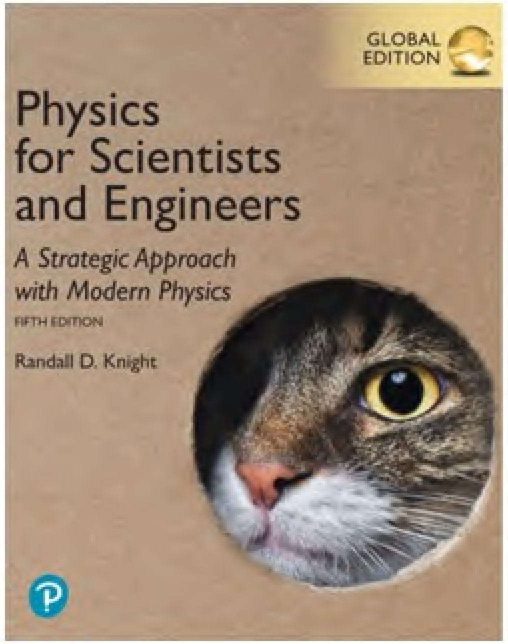
University Physics Volume 2
Samuel J. Ling, Jeff Sanny, William Moebs

P H Y S I C S For Scientists and Engineers, 5e, (2023)
Randall D. Knight









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Lecture 3:

Vector Algebra Part 3

- ❑ Kronecker Delta
- ❑ Levi-Civita symbol
- ❑ Expressing Vector Relations Using **Kronecker Delta** and **Levi-Civita Symbols**
- ❑ Relation Between Kronecker Delta and Levi-Civita Symbol
- ❑ Proof of the BAC-CAB Rule

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Leopold Kronecker
(1823 - 1891)

لئوپولد کرونکر
۱۲۰۱-۱۲۶۹

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

For Example,

$$\delta_{11} = 1$$

$$\delta_{22} = 1$$

$$\delta_{12} = 0$$

Tullio Levi-Civita
(1873-1941)

تولیو لوی چیویتا
(۱۲۵۱ - ۱۳۲۰)

$$\epsilon_{ijk} = \begin{cases} +1 & ijk = 123, 231, 312 \\ 0 & \text{If at least two of the indices are equal} \\ -1 & ijk = 132, 213, 321 \end{cases}$$

One simple algebraic form to express the Levi-Civita Symbol is given by:

$$\boxed{\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)}$$

Calculate the value of each of the following expressions

$$\sum_{i=1}^{3} \delta_{ii} = ?$$ **Answer: 3**

$$\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{mjk} = ?$$ **Answer: $2\delta_{im}$**

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\epsilon_{ijk} = ?$$ **Answer: 0**

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{ijk} = ?$$ **Answer: 6**

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$$\sum_{i=1}^{3} \delta_{ii} = ?$$

$$\boxed{\sum_{i=1}^{3} \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3}$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\epsilon_{ijk} = ?$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\epsilon_{ijk} = \sum_{i=1}^{3}\left(\delta_{i1}\epsilon_{i1k} + \delta_{i2}\epsilon_{i2k} + \delta_{i3}\epsilon_{i3k}\right)$$

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\epsilon_{ijk} &= \left(\delta_{11}\cancelto{0}{\epsilon_{11k}} + \cancelto{0}{\delta_{12}}\epsilon_{12k} + \cancelto{0}{\delta_{13}}\epsilon_{13k}\right)\\ &+ \left(\cancelto{0}{\delta_{21}}\epsilon_{21k} + \delta_{22}\cancelto{0}{\epsilon_{22k}} + \cancelto{0}{\delta_{23}}\epsilon_{23k}\right)\\ &+ \left(\cancelto{0}{\delta_{31}}\epsilon_{i1k} + \cancelto{0}{\delta_{32}}\epsilon_{32k} + \delta_{33}\cancelto{0}{\epsilon_{33k}}\right)\end{aligned}$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\epsilon_{ijk} = 0$$

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$$\boxed{\begin{aligned}\boldsymbol{A} &= A_x\hat{\boldsymbol{e}}_x + A_y\hat{\boldsymbol{e}}_y + A_z\hat{\boldsymbol{e}}_z \\ &= A_1\hat{\boldsymbol{e}}_1 + A_2\hat{\boldsymbol{e}}_2 + A_3\hat{\boldsymbol{e}}_3 \\ &= \sum_{i=1}^{3} A_i\hat{\boldsymbol{e}}_i\end{aligned}}$$

$$\boxed{\begin{aligned}\hat{\boldsymbol{e}}_x \cdot \hat{\boldsymbol{e}}_x = \hat{\boldsymbol{e}}_y \cdot \hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_z \cdot \hat{\boldsymbol{e}}_z = 1 \\ \hat{\boldsymbol{e}}_x \cdot \hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_x \cdot \hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}_y \cdot \hat{\boldsymbol{e}}_z = 0\end{aligned}}$$

$$\boxed{\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}}$$

$$\begin{aligned}\boldsymbol{A} \cdot \boldsymbol{B} &= \left(\sum_{i=1}^{3} A_i \hat{\boldsymbol{e}}_i\right) \cdot \left(\sum_{j=1}^{3} B_j \hat{\boldsymbol{e}}_j\right) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} A_i B_j \delta_{ij} \\ &= \sum_{i=1}^{3} A_i B_i\end{aligned}$$

$$\boxed{\boldsymbol{A} \times \boldsymbol{B} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} A_i B_j \hat{\boldsymbol{e}}_k}$$

$$\boxed{\left(\boldsymbol{A} \times \boldsymbol{B}\right)_k = \sum_{i=1}^{3}\sum_{j=1}^{3} \epsilon_{ijk} A_i B_j}$$

$$\boxed{\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j = \sum_{k=1}^{3} \epsilon_{ijk} \hat{\boldsymbol{e}}_k}$$

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$$\boxed{\left(\boldsymbol{A}\times\boldsymbol{B}\right)_k=\sum_{i=1}^{3}\sum_{j=1}^{3}\epsilon_{ijk}A_iB_j}$$

$k=3$

$$\left(\boldsymbol{A}\times\boldsymbol{B}\right)_3=\sum_{i=1}^{3}\sum_{j=1}^{3}\epsilon_{ij3}A_iB_j=\sum_{i=1}^{3}\left(\epsilon_{i13}A_iB_1+\epsilon_{i23}A_iB_2+\overset{0}{\epsilon_{i33}}A_iB_3\right)$$

$$\left(\boldsymbol{A}\times\boldsymbol{B}\right)_3=\left(\overset{0}{\epsilon_{113}}A_1B_1+\overset{1}{\epsilon_{123}}A_1B_2\right)+\left(\overset{-1}{\epsilon_{213}}A_2B_1+\overset{0}{\epsilon_{223}}A_2B_2\right)+\left(\overset{0}{\epsilon_{313}}A_3B_1+\overset{0}{\epsilon_{323}}A_3B_2\right)$$

$$\left(\boldsymbol{A}\times\boldsymbol{B}\right)_3=A_1B_2-A_2B_1$$

$$\left(\boldsymbol{A}\times\boldsymbol{B}\right)_z=A_xB_y-A_yB_x$$

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$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{im}(\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})$$

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

This relation is particularly useful in calculations involving cross products and determinants

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$$\begin{aligned}\left[\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})\right]_i &= \sum_j\sum_k \epsilon_{ijk}A_j(\boldsymbol{B}\times\boldsymbol{C})_k = \\ &= \sum_j\sum_k \epsilon_{ijk}A_j\sum_m\sum_l \epsilon_{klm}B_lC_m \\ &= \sum_j\sum_k\sum_m\sum_l \epsilon_{klm}\epsilon_{ijk}A_jB_lC_m\end{aligned}$$

We use the identity: $$\sum_k \epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\begin{aligned}\left[\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})\right]_i &= \sum_j\sum_m\sum_l\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)A_jB_lC_m \\ &= \sum_j(A_jC_j)B_i - \sum_j(A_jB_j)C_i \\ &= (\boldsymbol{A}\cdot\boldsymbol{C})B_i - (\boldsymbol{A}\cdot\boldsymbol{B})C_i\end{aligned}$$

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شاد و مهربان باشید

Be happy and kind