Kharazmi University

Faculty of Physics

Solutions for Homework 3

یادآوری: نماد کرونکر δ_{ij} و نماد لِوی-چیویتا ϵ_{ijk} به شکل زیر تعریف می‌شوند:

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} 1 & ijk = 123,\ 231,\ 312 \\ 0 & \text{در صورتی که حداقل دو اندیس برابر باشند} \\ -1 & ijk = 132,\ 213,\ 321 \end{cases}$$

می‌توان نماد لوی‌چیویتا را به شکل ساده‌ی زیر نیز بیان کرد:

$$\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$$

Problem 1:

Calculate the value of each of the following expressions.

☞ (a) $\sum_{i=1}^{3} \delta_{ii} = ?$

☞ (b) $\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\epsilon_{ijk} = ?$

☞ (c) $\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{mjk} = ?$

☞ (d) $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{ijk} = ?$

Answer Problem 1: See the lecture 3.

Problem 2:

Show that

$$\hat{\boldsymbol{e}}_i \cdot (\hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k) = \epsilon_{ijk}$$

where $\hat{\boldsymbol{e}}_n$ $(n = 1, 2, 3)$ are the orthonormal right-handed basis vectors in the order $1, 2, 3$.

Answer Problem 2: See the lecture 3.

Problem 3:

Show that

$$\boldsymbol{A}\times\boldsymbol{B}=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}A_iB_j\hat{\boldsymbol{e}}_k$$

Answer Problem 3:

$$\boldsymbol{A}=A_1\hat{\boldsymbol{e}}_1+A_2\hat{\boldsymbol{e}}_2+A_3\hat{\boldsymbol{e}}_3=\sum_{i=1}^{3}A_i\hat{\boldsymbol{e}}_i$$

$$\boldsymbol{B}=B_1\hat{\boldsymbol{e}}_1+B_2\hat{\boldsymbol{e}}_2+B_3\hat{\boldsymbol{e}}_3=\sum_{i=1}^{3}B_j\hat{\boldsymbol{e}}_j$$

$$\begin{aligned}\boldsymbol{A}\times\boldsymbol{B}&=\sum_{i=1}^{3}\sum_{j=1}^{3}A_iB_j\hat{\boldsymbol{e}}_i\times\hat{\boldsymbol{e}}_j\\&=\sum_{i=1}^{3}\sum_{j=1}^{3}A_iB_j\sum_{k=1}^{3}\epsilon_{ijk}\hat{\boldsymbol{e}}_k\\&=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}A_iB_j\hat{\boldsymbol{e}}_k\end{aligned}$$

Problem 4:

Show that

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}A_iB_jC_k$$

Answer Problem 4:

$$\begin{aligned}\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})&=\sum_{i=1}^{3}A_i(\boldsymbol{B}\times\boldsymbol{C})_i\\&=\sum_{i=1}^{3}A_i\sum_{i=j}^{3}\sum_{k=1}^{3}\epsilon_{jki}B_jC_k\\&=\sum_{i=1}^{3}\sum_{i=j}^{3}\sum_{k=1}^{3}\epsilon_{jki}A_iB_jC_k\\&=\sum_{i=1}^{3}\sum_{i=j}^{3}\sum_{k=1}^{3}\epsilon_{ijk}A_iB_jC_k\end{aligned}$$

Problem 5:

Show that

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}$$

Answer Problem 5:

$$\begin{aligned}
\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) &= \sum_{i=1}^{3} A_i(\boldsymbol{B}\times\boldsymbol{C})_i \\
&= \sum_{i=1}^{3} A_i \sum_{i=j}^{3}\sum_{k=1}^{3} \epsilon_{jki} B_j C_k \\
&= \sum_{i=1}^{3}\sum_{i=j}^{3}\sum_{k=1}^{3} \epsilon_{jki} A_i B_j C_k \\
&= \sum_{k=1}^{3}\left(\sum_{i=1}^{3}\sum_{i=j}^{3} \epsilon_{ijk} A_i B_j\right) C_k \\
&= \sum_{k=1}^{3} (\boldsymbol{A}\times\boldsymbol{B})_k C_k \\
&= (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}
\end{aligned}$$

Problem 6:

Show that

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Answer Problem 6: The left side is a sum of three terms:

$$\epsilon_{jki}\epsilon_{jlm} = \epsilon_{1ki}\epsilon_{1lm} + \epsilon_{2ki}\epsilon_{2lm} + \epsilon_{3ki}\epsilon_{3lm}.$$

This identity can be understood as follows: For the Levi-Civita symbol to have nonzero values, all indices must differ in each factor on the left side ($j \neq k \neq i$ and $j \neq l \neq m$). Since the first index j is shared in both Levi-Civita symbols, and indices take values 1, 2, or 3, there are two possibilities: either $k = l$ or $k = m$.

- If $k = l$, the remaining indices must satisfy $i = m$, resulting in a term $\delta_{kl}\delta_{im}$.
- If $k = m$, swapping l and m introduces a minus sign, leading to $-\delta_{km}\delta_{il}$.

Combining these cases, the full identity becomes:

$$\epsilon_{jki}\epsilon_{jlm} = \delta_{kl}\delta_{im} - \delta_{km}\delta_{il}.$$

Proper attention to index ordering ensures the correct signs in the final expression.

Problem 7:

Show that

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijkmjk} = 2\delta_{im}$$

Answer Problem 7:

Using

$$\sum_{i=1}^{3}\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

we can write:

$$\begin{aligned}
\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\epsilon_{mjk} &= \sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{kij}\epsilon_{kmj} \\
&= \sum_{j=1}^{3}(\delta_{im}\delta_{jj} - \delta_{ij}\delta_{jm}) \\
&= \sum_{j=1}^{3}(\delta_{im} - \delta_{ij}\delta_{jm}) \\
&= \sum_{j=1}^{3}\delta_{im} - \sum_{j=1}^{3}\delta_{ij}\delta_{jm} \\
&= 3\delta_{im} - \delta_{im} \\
&= 2\delta_{im}
\end{aligned}$$

Problem 8:

Show that $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$. This is known as the BAC-CAB rule.

Answer Problem 8:

$$\begin{aligned}
[\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})]_i &= \sum_j\sum_k\epsilon_{ijk}A_j(\boldsymbol{B}\times\boldsymbol{C})_k \\
&= \sum_j\sum_k\epsilon_{ijk}A_j\sum_m\sum_l\epsilon_{klm}B_lC_m \\
&= \sum_j\sum_k\sum_m\sum_l\epsilon_{klm}\epsilon_{ijk}A_jB_lC_m
\end{aligned}$$

using

$$\sum_k\epsilon_{klm}\epsilon_{ijk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$
\begin{aligned}
[\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})]_i &= \sum_j \sum_m \sum_l (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m \\
&= \sum_j \sum_m \sum_l \delta_{il}\delta_{jm} A_j B_l C_m - \sum_j \sum_m \sum_l \delta_{im}\delta_{jl} A_j B_l C_m \\
&= \sum_j \sum_l \delta_{il} A_j B_l C_j - \sum_j \sum_m \delta_{im} A_j B_j C_m \\
&= \sum_j (A_j C_j) B_i - \sum_j (A_j B_j) C_i \\
&= (\boldsymbol{A} \cdot \boldsymbol{C}) B_i - (\boldsymbol{A} \cdot \boldsymbol{B}) C_i
\end{aligned}
$$

Therefor,

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$