Fundamentals of Physics II

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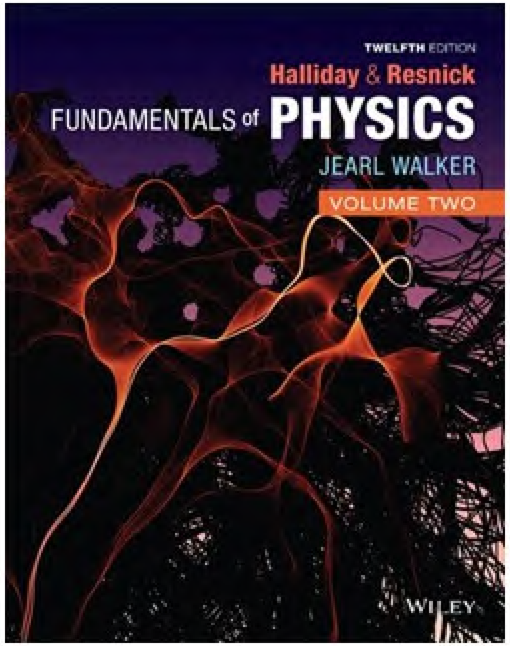
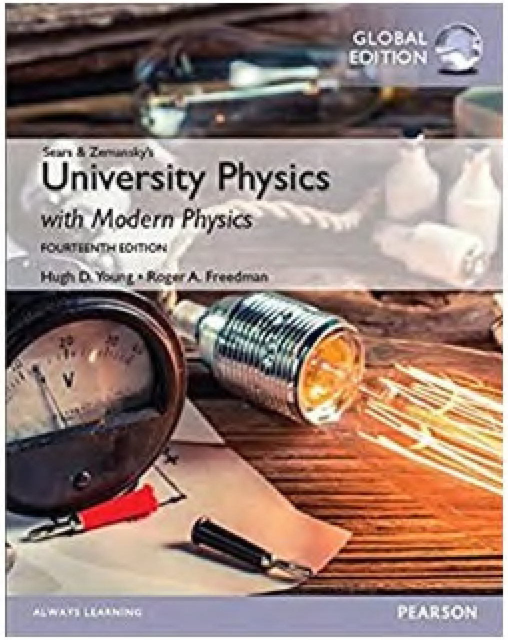
اگر همواره مانند گذشته بیندیشید، همیشه همان چیزهایی را به‌دست می‌آورید که تاکنون کسب کرده‌اید

If you always think the way you've always thought, you'll always get what you've always got.

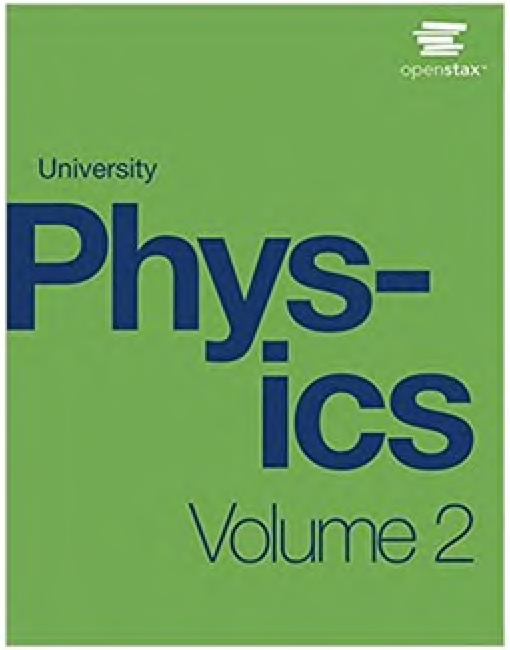
Fundamentals of Physics (12th Ed.)
Halliday, David; Resnick, Robert; Walker, Jearl



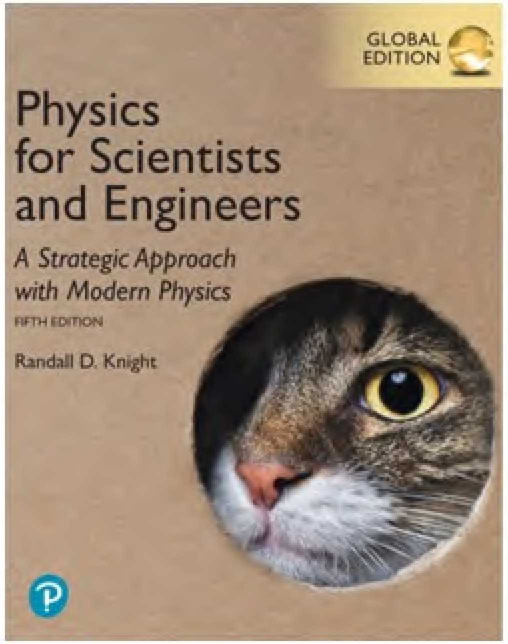
University Physics with Modern Physics (14th Global Ed.)
Hugh D. Young, Roger A. Freedman



University Physics Volume 2
Samuel J. Ling, Jeff Sanny, William Moebs



P H Y S I C S For Scientists and Engineers, 5e, (2023)
Randall D. Knight



Lecture 2:

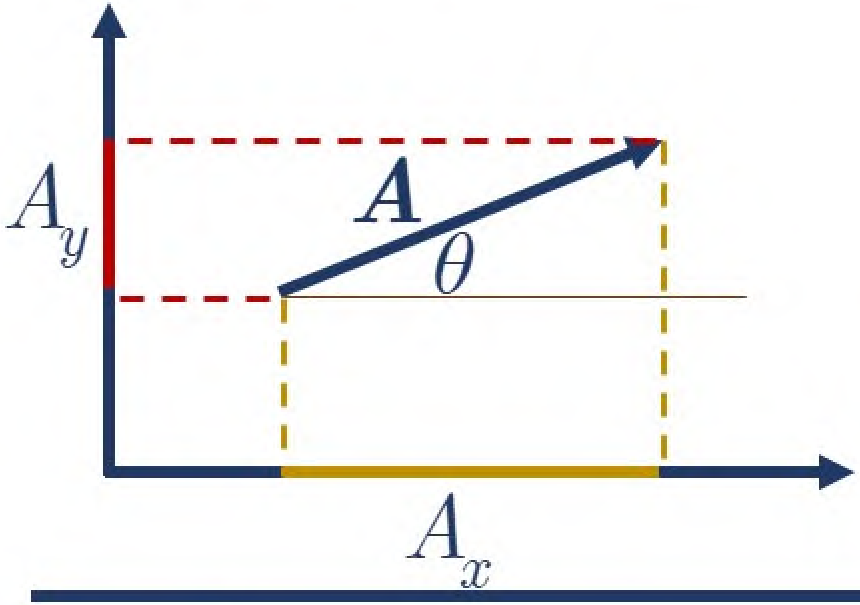
Vector Algebra Part 2

دانشگاه خوارزمی

- ❑ Representation of a vector in terms of its components
- ❑ **Analytical method of vector addition**
- ❑ **Dot product and vector product in terms of components**
- ❑ **The angle between two vectors in terms of their components**
- ❑ **Triple Product**
- ❑ **Vector Division!!!!**

you cannot divide a vector by a vector

دانشگاه خوارزمی

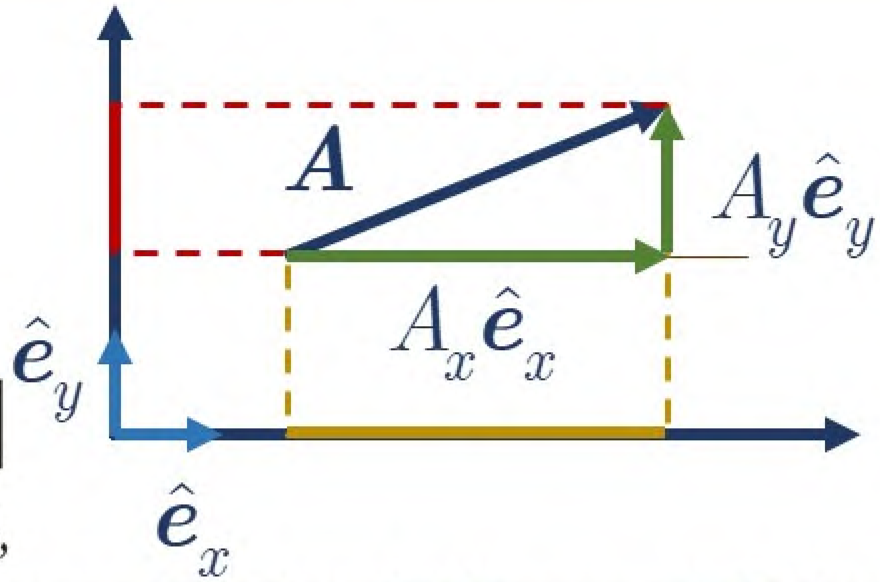


$$A^2 = A_x^2 + A_y^2$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

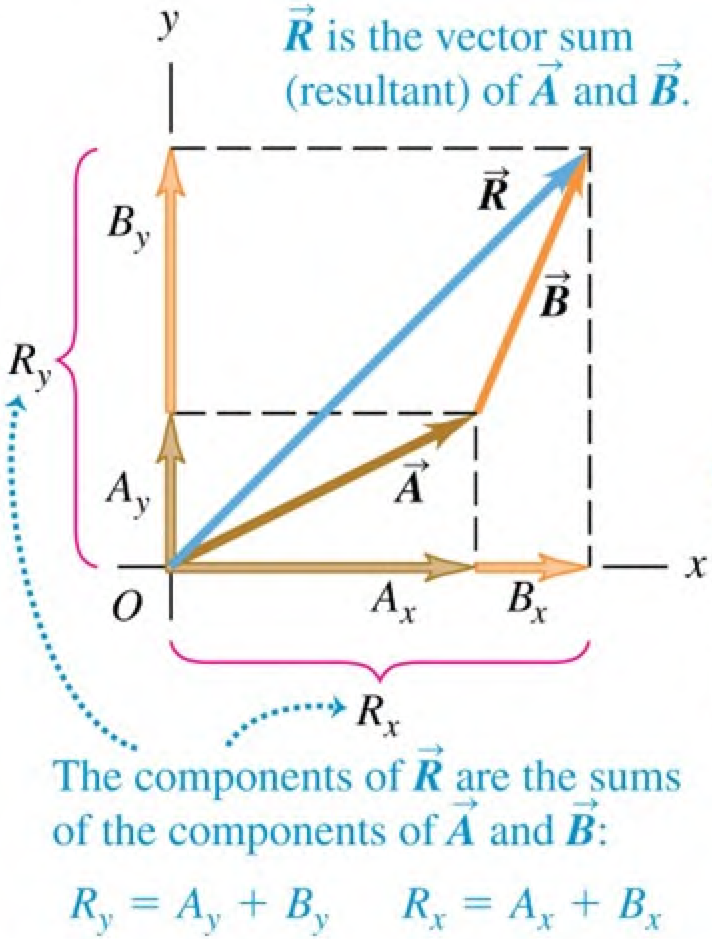
$$A_x = A\cos\theta$$

$$A_y = A\sin\theta$$



$$\boldsymbol{A} = A_x\hat{\boldsymbol{e}}_x + A_y\hat{\boldsymbol{e}}_y$$

$$\boldsymbol{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$



Two Dimensional

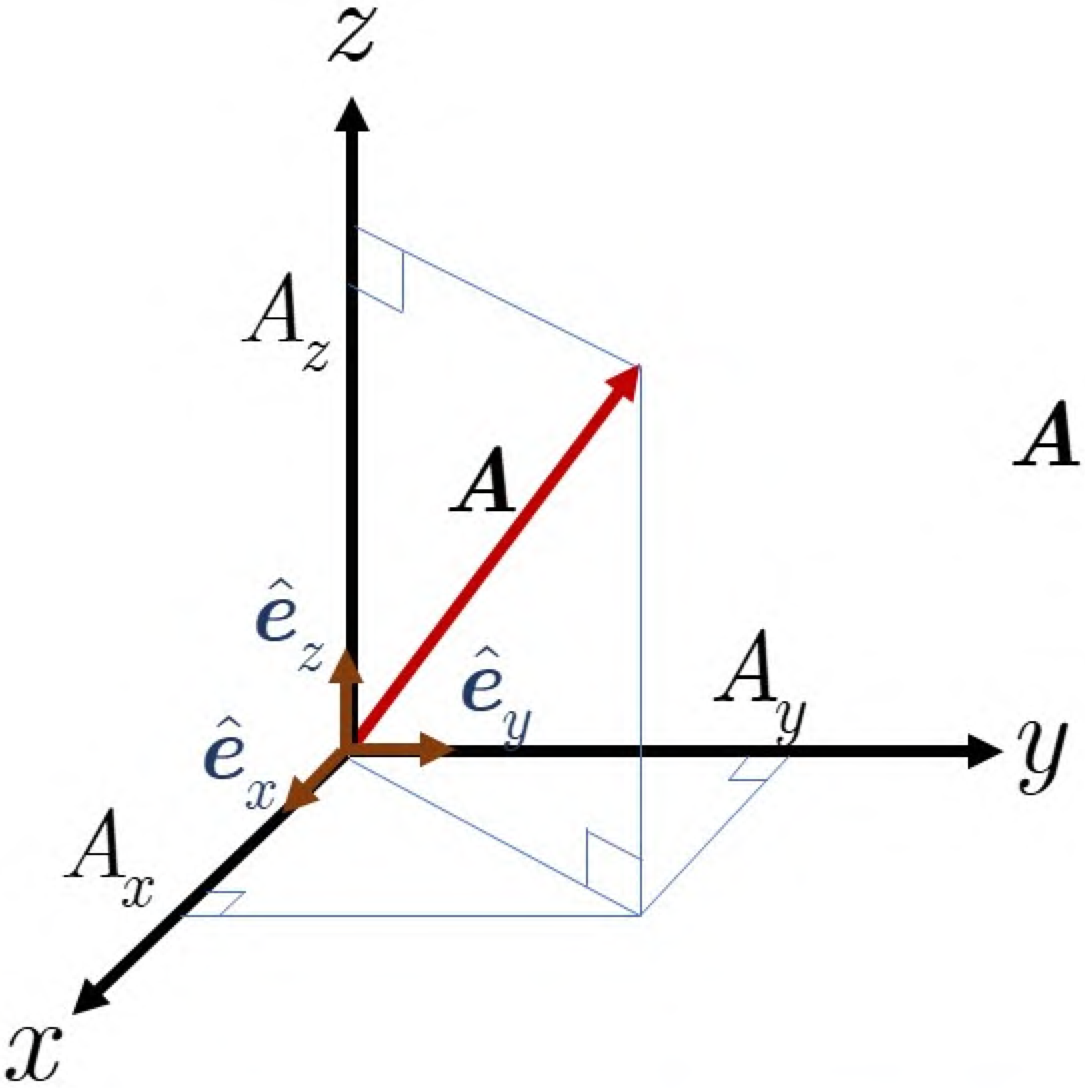
$$\boldsymbol{A} + \boldsymbol{B} = \left(A_x + B_x\right)\hat{\boldsymbol{e}}_x + \left(A_y + B_y\right)\hat{\boldsymbol{e}}_y$$

Three Dimensional

$$\boldsymbol{A} + \boldsymbol{B} = \left(A_x + B_x\right)\hat{\boldsymbol{e}}_x + \left(A_y + B_y\right)\hat{\boldsymbol{e}}_y + \left(A_z + B_z\right)\hat{\boldsymbol{e}}_z$$

Proof:

$$\alpha\boldsymbol{A} = \left(\alpha A_x\right)\hat{\boldsymbol{e}}_x + \left(\alpha A_y\right)\hat{\boldsymbol{e}}_y + \left(\alpha A_z\right)\hat{\boldsymbol{e}}_z$$



$$\boldsymbol{A} = A_x\hat{\boldsymbol{e}}_x + A_y\hat{\boldsymbol{e}}_y + A_z\hat{\boldsymbol{e}}_z$$

$$\hat{\boldsymbol{e}}_x \cdot \hat{\boldsymbol{e}}_y = \left|\hat{\boldsymbol{e}}_x\right|\left|\hat{\boldsymbol{e}}_y\right| \cos\frac{\pi}{2} = 0$$

$$\hat{\boldsymbol{e}}_x \cdot \hat{\boldsymbol{e}}_x = \hat{\boldsymbol{e}}_y \cdot \hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_z \cdot \hat{\boldsymbol{e}}_z = 1$$
$$\hat{\boldsymbol{e}}_x \cdot \hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_x \cdot \hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}_y \cdot \hat{\boldsymbol{e}}_z = 0$$

$$\begin{aligned} \boldsymbol{A} \cdot \hat{\boldsymbol{e}}_x &= \left(A_x \hat{\boldsymbol{e}}_x + A_y \hat{\boldsymbol{e}}_y\right) \cdot \hat{\boldsymbol{e}}_x \\ &= A_x \hat{\boldsymbol{e}}_x \cdot \hat{\boldsymbol{e}}_x + A_y \hat{\boldsymbol{e}}_y \cdot \hat{\boldsymbol{e}}_x \\ &= A_x \end{aligned}$$

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= \boldsymbol{A} \cdot \left(B_x \hat{\boldsymbol{e}}_x + B_y \hat{\boldsymbol{e}}_y\right) \\ &= B_x \boldsymbol{A} \cdot \hat{\boldsymbol{e}}_x + B_y \boldsymbol{A} \cdot \hat{\boldsymbol{e}}_y \\ &= A_x B_x + A_y B_y \end{aligned}$$

Three Dimensional $\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$

$$\boldsymbol{A} \cdot \boldsymbol{B} = AB\cos\theta = A_x B_x + A_y B_y + A_z B_z$$

$$\boldsymbol{A} \cdot \boldsymbol{A} = A^2 = A_x^2 + A_y^2 + A_z^2$$

$$\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{AB}$$

$$\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

دانشگاه خوارزمی

Let $\boldsymbol{A} = 2\hat{e}_x$ and $\boldsymbol{B} = \sqrt{3}\hat{e}_x + \hat{e}_y$. Find the angle between them.

$$\cos\theta = \frac{A_x B_x + A_y B_y}{\sqrt{A_x^2 + A_y^2}\sqrt{B_x^2 + B_y^2}} = \frac{\sqrt{3}}{2}$$

$$\theta = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

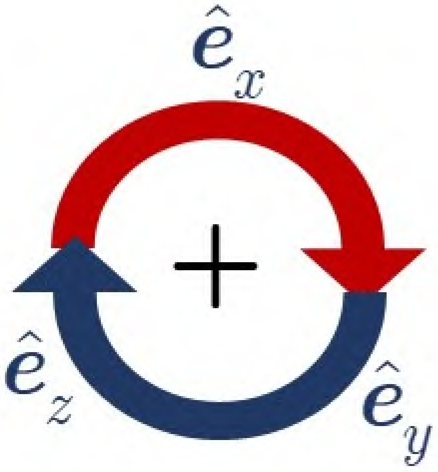
دانشگاه خوارزمی

Write a program (preferably in Python) that asks the user for the components of two vectors. Then, given the components, calculates the angle between the two vectors in radians and degrees.

$$\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_x = \hat{\boldsymbol{e}}_y \times \hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_z = 0$$

$$\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_z$$

$$\hat{\boldsymbol{e}}_y \times \hat{\boldsymbol{e}}_x = -\hat{\boldsymbol{e}}_z$$



$$\begin{aligned}\boldsymbol{A}\times\hat{\boldsymbol{e}}_x &= \left(A_x\hat{\boldsymbol{e}}_x + A_y\hat{\boldsymbol{e}}_y + A_z\hat{\boldsymbol{e}}_z\right)\times\hat{\boldsymbol{e}}_x\\ &= 0 + A_y\hat{\boldsymbol{e}}_y\times\hat{\boldsymbol{e}}_x + A_z\hat{\boldsymbol{e}}_z\times\hat{\boldsymbol{e}}_x\\ &= -A_y\hat{\boldsymbol{e}}_z + A_z\hat{\boldsymbol{e}}_y\end{aligned}$$

$$\begin{aligned}\boldsymbol{A}\times\hat{\boldsymbol{e}}_y &= \left(A_x\hat{\boldsymbol{e}}_x + A_y\hat{\boldsymbol{e}}_y + A_z\hat{\boldsymbol{e}}_z\right)\times\hat{\boldsymbol{e}}_y\\ &= A_x\hat{\boldsymbol{e}}_x\times\hat{\boldsymbol{e}}_y + 0 + A_z\hat{\boldsymbol{e}}_z\times\hat{\boldsymbol{e}}_y\\ &= A_x\hat{\boldsymbol{e}}_z - A_z\hat{\boldsymbol{e}}_x\end{aligned}$$

$$\begin{aligned}\boldsymbol{A}\times\hat{\boldsymbol{e}}_z &= \left(A_x\hat{\boldsymbol{e}}_x + A_y\hat{\boldsymbol{e}}_y + A_z\hat{\boldsymbol{e}}_z\right)\times\hat{\boldsymbol{e}}_z\\ &= A_x\hat{\boldsymbol{e}}_x\times\hat{\boldsymbol{e}}_z + A_y\hat{\boldsymbol{e}}_y\times\hat{\boldsymbol{e}}_z + 0\\ &= -A_x\hat{\boldsymbol{e}}_y + A_y\hat{\boldsymbol{e}}_x\end{aligned}$$

$$\boldsymbol{A}\times\boldsymbol{B} = \boldsymbol{A}\times\left(B_x\hat{\boldsymbol{e}}_x + B_y\hat{\boldsymbol{e}}_y + B_z\hat{\boldsymbol{e}}_z\right) = B_x\boldsymbol{A}\times\hat{\boldsymbol{e}}_x + B_y\boldsymbol{A}\times\hat{\boldsymbol{e}}_y + B_z\boldsymbol{A}\times\hat{\boldsymbol{e}}_z$$
$$= B_x\left(-A_y\hat{\boldsymbol{e}}_z + A_z\hat{\boldsymbol{e}}_y\right) + B_y\left(A_x\hat{\boldsymbol{e}}_z - A_z\hat{\boldsymbol{e}}_x\right) + B_z\left(-A_x\hat{\boldsymbol{e}}_y + A_y\hat{\boldsymbol{e}}_x\right)$$

$$\begin{aligned}\boldsymbol{A}\times\boldsymbol{B} &= \left(A_yB_z - A_zB_y\right)\hat{\boldsymbol{e}}_x \\ &\quad -\left(A_xB_z - A_zB_x\right)\hat{\boldsymbol{e}}_y \\ &\quad +\left(A_xB_y - A_yB_x\right)\hat{\boldsymbol{e}}_z\end{aligned}$$

$$\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix}\hat{\boldsymbol{e}}_x - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix}\hat{\boldsymbol{e}}_y + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}\hat{\boldsymbol{e}}_z$$

$$\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$\boldsymbol{F} = -8\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ **Newtons acts on a particle with position vector** $\boldsymbol{r} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ **. What are**

(i) the torque on the particle about the origin and

(ii) the angle between the direction of $\boldsymbol{F}$ and $\boldsymbol{r}$?

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = (3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}) \times (-8\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}) = 50\hat{\boldsymbol{k}}\ \text{N.m}$$

$$\cos\theta = \frac{\boldsymbol{r} \cdot \boldsymbol{F}}{rF} = \frac{(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}) \cdot (-8\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}})}{\sqrt{9+16}\sqrt{64+36}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$$\boldsymbol{B}\times\boldsymbol{C}=\begin{vmatrix}\hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

$$\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C}=?$$

$$\boldsymbol{A}\times\left(\boldsymbol{B}\times\boldsymbol{C}\right)=?$$

$$\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C}=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

$$\boldsymbol{A}\times\left(\boldsymbol{B}\times\boldsymbol{C}\right)=\boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C})-\boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$$ **BAC-CAB rule**

$$\boxed{\boldsymbol{A}\times\left(\boldsymbol{B}\times\boldsymbol{C}\right)\neq\left(\boldsymbol{A}\times\boldsymbol{B}\right)\times\boldsymbol{C}}$$

دانشگاه خوارزمی

$$\boldsymbol{A}\cdot\left(\boldsymbol{B}\times\boldsymbol{C}\right)=\boldsymbol{B}\cdot\left(\boldsymbol{C}\times\boldsymbol{A}\right)$$

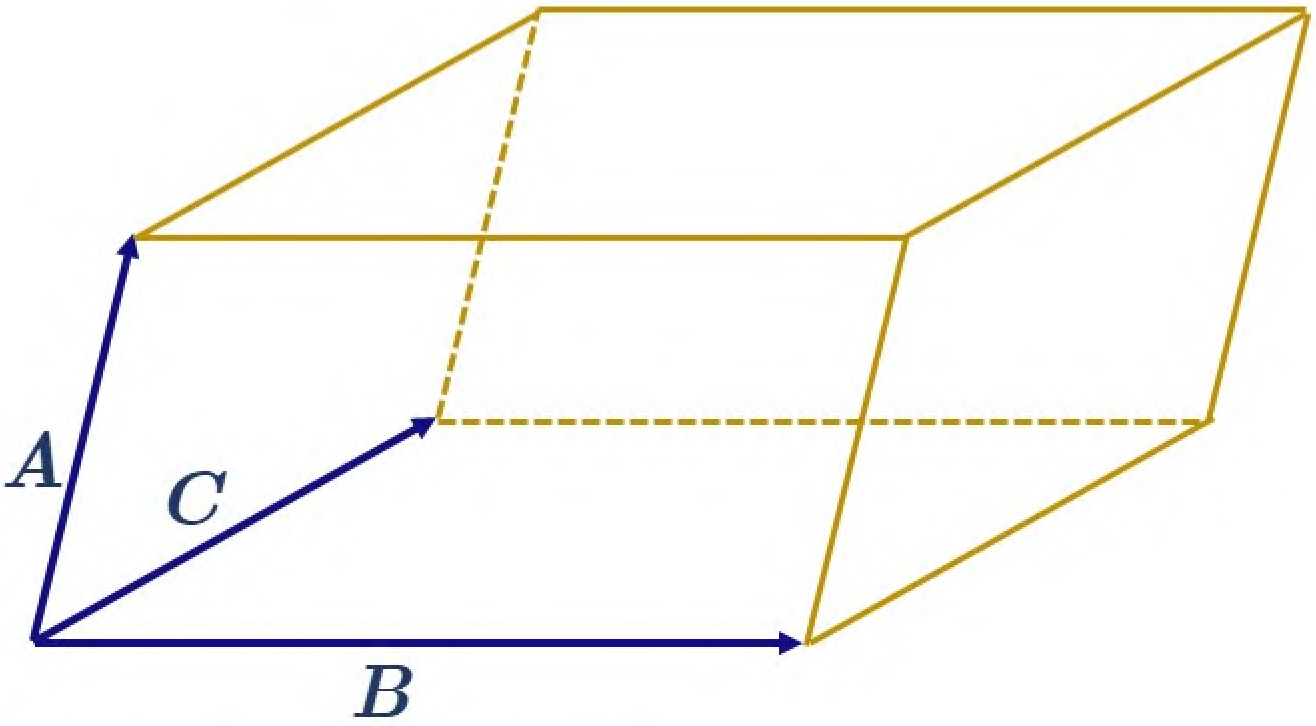
$$=\boldsymbol{C}\cdot\left(\boldsymbol{A}\times\boldsymbol{B}\right)$$

$$=-\boldsymbol{A}\cdot\left(\boldsymbol{C}\times\boldsymbol{B}\right)$$

$$=-\boldsymbol{B}\cdot\left(\boldsymbol{A}\times\boldsymbol{C}\right)$$

$$=-\boldsymbol{C}\cdot\left(\boldsymbol{B}\times\boldsymbol{A}\right)$$

دانشگاه خوارزمی



$$V = \left| \boldsymbol{A} \cdot \left(\boldsymbol{B} \times \boldsymbol{C} \right) \right|$$ **Volume of a parallelepiped**

In the realm of real numbers, we often encounter equations such as $3x = 6$, where x represents an unknown quantity, we aim to determine. Solving this equation, we find a unique solution: $x = 6/3 = 2$. This demonstrates the operation of division, which is well-defined and meaningful within the real number system. The question then arises: can we define and apply division in an analogous way for vectors?

Suppose we have a vector equation of this form:

$$\boldsymbol{A} \cdot \boldsymbol{X} = a$$

It is easy to check that x can be written as:

$$\boldsymbol{X} = \frac{\boldsymbol{A}}{\boldsymbol{A} \cdot \boldsymbol{A}} a \mathbb{S}$$

This answer is not unique

$$\boldsymbol{X} = \frac{\boldsymbol{A}}{\boldsymbol{A} \cdot \boldsymbol{A}} a + \boldsymbol{B} \quad ; \qquad \boldsymbol{A} \cdot \boldsymbol{B} = 0$$

Also consider another vector equation of this form: $\boldsymbol{A} \times \boldsymbol{X} = \boldsymbol{C}$

The solution to this equation is also not unique

$$\boldsymbol{X} = \frac{\boldsymbol{C} \times \boldsymbol{A}}{\boldsymbol{A} \cdot \boldsymbol{A}} + k\boldsymbol{A} \qquad k \text{ is an arbitrary value}$$

Suppose we have a system of equations in the following form:

$$\begin{cases} \boldsymbol{A} \cdot \boldsymbol{X} = a \\ \boldsymbol{A} \times \boldsymbol{X} = \boldsymbol{C} \end{cases}$$

The unique solution of this system of equations is

$$\boldsymbol{X} = \frac{\boldsymbol{C} \times \boldsymbol{A}}{\boldsymbol{A} \cdot \boldsymbol{A}} + \frac{a}{\boldsymbol{A} \cdot \boldsymbol{A}} \boldsymbol{A}$$

شاد و مهربان باشید

Be happy and kind

دانشگاه خوارزمی